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Note

Graphs with not all possible path-kernels

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Abstract

The Path Partition Conjecture states that the vertices of a graph G with longest path of length c may be partitioned into two parts X and Y such that the longest path in the subgraph of G induced by X has length at most a and the longest path in the subgraph of G induced by Y has length at most b, where $a+b=c$. Moreover, for each pair a,b such that $a+b=c$ there is a partition with this property. A stronger conjecture by Broere, Hajnal and Mihók, raised as a problem by Mihók in 1985, states the following: For every graph G and each integer k, $c \geqslant k \geqslant 2$ there is a partition of $V(G)$ into two parts $(K,\overline{K})$ such that the subgraph $G[K]$ of G induced by K has no path on more than $k-1$ vertices and each vertex in $\overline{K}$ is adjacent to an endvertex of a path on $k-1$ vertices in $G[K]$. In this paper we provide a counterexample to this conjecture.

Keywords: Path Partition Conjecture; P_k-kernel; Path-kernel

1. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $X \subseteq V(G)$, we denote the subgraph of G induced by X by $G[X]$. We denote by $\tau(G)$, the number of vertices in a longest path in G. For any graph G, a subset K of $V(G)$ is called a *P_k-kernel* of G if

(1) $\tau(G[K]) \leqslant k-1$ and
(2) every vertex $v \in V(G)\backslash K$ is adjacent to an endvertex of a path on $k-1$ vertices in $G[K]$.

If $K \subseteq V(G)$ is a P_k-kernel of G, we call $V(G)\backslash K$ a *co-P_k-kernel.*

Based on a problem raised in 1985 by Mihók [6], Broere et al. [1] conjecture that every graph has a P_k-kernel for every integer $k \geqslant 2$. This conjecture has been verified for small values of k (see [2,4]).

Mihók's conjecture was inspired by a weaker conjecture due to Laborde et al. [5] which proposes that the vertices of a graph G with longest path on c vertices may be partitioned into two parts X and Y such that the longest path in $G[X]$ has at most a vertices and the longest path in $G[Y]$ has at most b vertices, where $a+b=c$. Moreover, for each pair a,b such that $a+b=c$ there is a partition with this property. This conjecture has become known as the Path Partition Conjecture.

In order to derive the Path Partition Conjecture from Mihók's conjecture it suffices to put k equal to either $a+1$ or $b+1$. Thus it suffices to verify Mihók's conjecture for $k \leqslant c/2+1$. A particularly interesting case of both conjectures occurs when $a=b=k$.

Dunbar and Frick [3] introduced the following conjecture which is stronger than the Path Partition Conjecture: If M is a maximum P_{n+1}-free set of vertices of G, with $n < \tau(G)$, then $\tau(G-M) \leqslant \tau(G)-n$.

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This conjecture is false. To see this, consider a hypotraceable graph with $n+2$ vertices. (A graph G is *hypotraceable* if it has no Hamiltonian path but every vertex-deleted subgraph $G-x$ does.) Then $\tau(G)=n+1$. Any set of n vertices is a maximum P_{n+1}-free set of vertices of G. The complement may have an edge and contradicts therefore the conjecture. However, in order to derive the Path Partition Conjecture, it suffices to verify the conjecture of Dunbar and Frick for $n \leqslant \tau(G)/2$ and we do not have counterexamples for that.

Here we shall consider Mihók's conjecture for large values of k, specifically $k=\tau(G)$. Such a co-P_k-kernel is an independent set I such that the longest path of $G-I$ is smaller than that of G, and moreover, for every vertex x in I, G has a longest path starting at x and containing no other vertex of I. The analogous problem for longest directed paths in acyclic directed graphs is trivial: Just let I consist of all initial vertices of the longest directed paths. But, this special case of Mihók's conjecture is false, as we show.

It seems natural to try first the hypotraceable graphs. However, all the hypotraceable graphs with n vertices described in [7] have a co-P_{n-1}-kernel consisting of just two vertices. So, it seems that a different construction is needed for counterexamples to Mihók's conjecture. Such a construction is presented below.

2. The construction

2.1. The blobs

We refer to the graph in Fig. 1 as a *blob*. Note that we have presented a plane embedding of the blob. The outer face is a subdivision of the 4-cycle $acdb$ in which the edges ac and db have each been subdivided 20 times while the remaining two edges have each been subdivided 10 times. The structure in the interior of this outer cycle has 16 vertices. We shall use these actual numbers for concreteness although it will become clear that there is great flexibility within this general structure to produce the counterexample we seek. The blob shown has the following properties.

(1) All longest paths from a to b avoid the vertices Z and Z'. It is easily checked that, in fact, *the* longest path from a to b runs $a,\dots,c,\dots,d,\dots,b$. We shall denote this by $\overrightarrow{ab}$ and note that $\overrightarrow{ab}$ contains 54 vertices.

(2) All longest paths from a to c avoid the vertices Z and Z'. It is easily checked that, in fact, *the* longest path from a to c runs $a,\dots,b,\dots,d,\dots,c$. We shall denote this by $\overrightarrow{ac}$ and note that $\overrightarrow{ac}$ contains 44 vertices.

(3) Every longest path from b to c passes through both Z and Z'. Again, it is easily checked that *the* longest path from b to c runs $b,\dots,d,s,S,\dots,t,y,x,z,\ Z,u,\dots,V,v,w,a,\dots,c$. We shall denote this by $\overrightarrow{bc}$ and note that $\overrightarrow{bc}$ contains 58 vertices.

(4) Every longest path starting at c ends at either Z or Z'. Here again one can easily check that the longest paths from c run either $c,\dots,a,\dots,b,\dots,d,s,S,\dots,t,T,U,u,\dots,V,v,w,x,z,Z$ or $c,\dots,a,\dots,b,\dots,d,s,S,\dots,t,T,U,u,Z,z,x,w,v,V,\dots,Z'$. We shall denote these by $\overrightarrow{cZ}$, $\overrightarrow{cZ'}$ respectively and note that each contains 69 vertices. Note also that each of these paths has the same vertex set. When we wish to refer to a path as being either $\overrightarrow{cZ}$ or $\overrightarrow{cZ'}$ we use $\overrightarrow{c\zeta}$ with the understanding that $\zeta\in\{Z,Z'\}$.

(5) Every longest path starting at b ends at a neighbour of u. A straightforward check determines that a longest path from b runs $b,\dots,d,\dots,c,\dots,a,w,x,z,Z,u,\dots,V,v,s,S,\dots,t,T,U$ or in a similar fashion but following around the cycle in the opposite way from u. Both paths cover the same number of vertices; 69. Denote this by $\overrightarrow{b\beta}$.

(6) The maximum number of vertices contained in a pair of disjoint paths, P,Q where P joins a to either b or c and Q has the remaining vertex from $\{b,c\}$ as an endvertex is realized by P joining a to c and Q joining b to Z or Z'. Once again, it is easy to check that the only pairs of paths realizing the maximum vertex coverage is $P=a,\dots,c$ and $Q=b,\dots,d,s,S,\dots,t,T,U,u,\dots,V,v,w,x,z,Z$ (or $Q=b,\dots,d,s,S,\dots,t,T,U,u,Z,z,x,w,v,V,\dots,Z'$). We denote these by $a\to c$ and $\overrightarrow{bZ}$ ($\overrightarrow{bZ'}$) respectively and note that $a\to c$ contains 22 vertices while each of $\overrightarrow{bZ}$ and $\overrightarrow{bZ'}$ contains 37 vertices. As in Observation 4 above, we note that $\overrightarrow{bZ}$ and $\overrightarrow{bZ'}$ have exactly the same vertex sets and use $\overrightarrow{b\zeta}$ to denote the path Q with the understanding that $\zeta\in\{Z,Z'\}$.

In our later discussions we shall have cause to refer to the paths indicated above. When the direction along which the path is to be followed is the reverse of that given in the definition, we shall reverse the order of the letters. So, for example, if we were to follow the path $\overrightarrow{ab}$ in reverse (from b to a), we would write $\overrightarrow{ba}$.

2.2. The big picture

We construct a graph G as follows. Take two copies of K_4, each with a distinguished vertex, and add an edge joining the two distinguished vertices to form a new graph H. The six undistinguished vertices in H induce two triangles. Assign

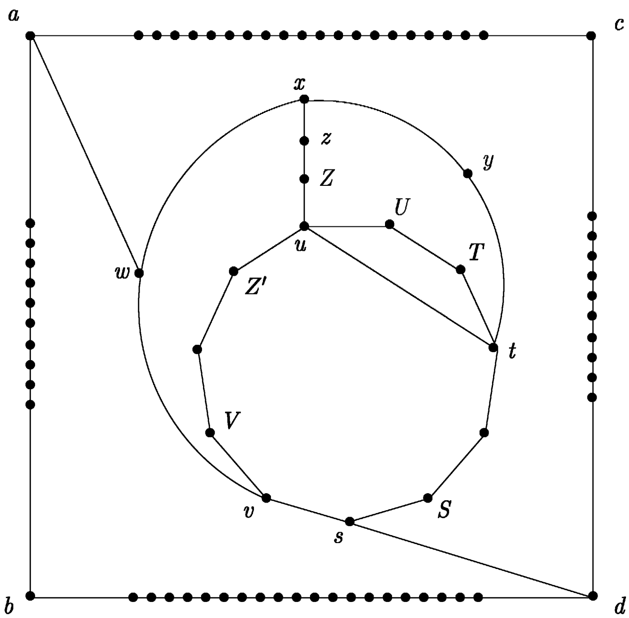

Fig. 1. A "blob".

an orientation to each triangle and replace each of the undistinguished vertices by a blob (see Fig. 1) so that in each blob, vertex a is adjacent to a distinguished vertex and vertex b is adjacent to vertex c in the succeeding blob around the triangle (see Fig. 2). To aid in the following analysis, we assume that the blobs in G are numbered from 1 to 6 with all evenly numbered blobs adjacent to the same distinguished vertex, call it A_0. All oddly numbered blobs are assumed to be adjacent to the other distinguished vertex, A_1. The vertices within each blob are then assumed to be labelled as in Fig. 1 but with subscripts denoting the number assigned to a given blob. Then, for example, we have b_1 adjacent to c_3, b_3 adjacent to c_5 and b_5 adjacent to c_1 and all of a_1, a_3 and a_5 are adjacent to A_1. Similarly, the path segments defined within a blob will now be referred to using this subscripted notation. For example, we will use $\overrightarrow{a_ib_i}$ to indicate the path corresponding to $\overrightarrow{ab}$ in the blob containing a_i.

Lemma 2.1. *If Π is a longest path in G, then Π has endvertices ζ_i and ζ_j with $i \in \{1,3,5\}$, $j \in \{2,4,6\}$ and $\zeta \in \{Z, Z'\}$. Moreover, there is a longest path in G for each such pair (ζ_i, ζ_j), it contains 364 vertices and the form of the path is:* $\Pi_{i,j} = \overrightarrow{\zeta_i c_i}, \overrightarrow{b_{i-2}c_{i-2}}, \overrightarrow{b_{i-4}a_{i-4}}, A_1, A_0, \overrightarrow{a_{j-4}b_{j-4}}, \overrightarrow{c_{j-2}b_{j-2}}, \overrightarrow{c_j\zeta_j}$.

Proof. Consider a path $\Pi_{i,j}$ as in the statement of the lemma. It is clear that such a path exists for each pair ζ_i, ζ_j so we need only show that $\Pi_{i,j}$ is a longest path in G.

Let us consider the structure of a longest path Π in G. By the symmetry of G, Π must have one endvertex in a blob adjacent to A_0 and the other in a blob adjacent to A_1. Indeed, the symmetry of G indicates that a longest path will cover precisely the same number of vertices in blobs adjacent to A_0 as in blobs adjacent to A_1. Moreover, since G has 482 vertices and $\Pi_{i,j}$ has 364 vertices, a longest path in G must visit all six blobs. We shall focus first on that segment of a longest path in G which goes from A_1 into the odd numbered blobs adjacent to it. Call this segment Π_{odd}. Without loss of generality, we may assume that Π_{odd} runs $A_1, a_1, \ldots$. Now, as observed above, Π_{odd} must visit all three blobs adjacent to A_1 so there are two possibilities: either Π_{odd} passes through the blob containing a_1 and never returns, or it passes through once, exiting at either b_1 or c_1, returning through the remaining vertex to terminate in this blob. In the latter case, Π_{odd} must transit each of the remaining blobs using either $\overrightarrow{b_3c_3}$ and $\overrightarrow{b_5c_5}$ or $\overrightarrow{c_5b_5}$ and $\overrightarrow{c_3b_3}$. Either way, these segments cover 116 vertices leaving us to cover at least 65 vertices in the blob containing a_1. But, as observed earlier, the maximum number of vertices we can cover in a blob with a pair of disjoint paths, one from a_1 to b_1 (c_1) and the other with c_1 (b_1) as an endvertex is 59. So we must pass through this first blob just once and terminate in a different blob. Thus Π_{odd} either has the form indicated in $\Pi_{i,j}$ or $\Pi_{\text{odd}} = A_1, \overrightarrow{a_1c_1}, \overrightarrow{b_5c_5}, \overrightarrow{b_3\beta_3}$. Π_{odd} so formed covers $1 + 44 + 58 + 69 = 172 < 182$ vertices. The result now follows. $\square$

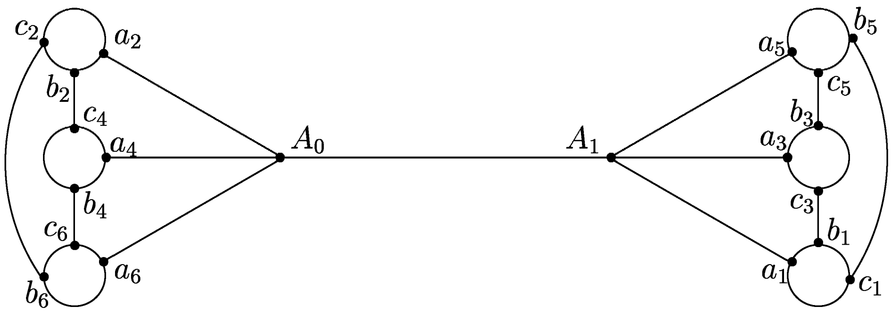

Fig. 2. The graph G.

3. The main result

Theorem 3.1. *The graph G contains no P_{364}-kernel.*

Proof. Let us assume to the contrary that G does have a P_{364}-kernel, K. Let $\overline{K} = V(G)\backslash K$ be the co-P_{364}-kernel corresponding to K. Now, since G contains paths covering 364 vertices, $K \neq V(G)$ and $\overline{K} \neq \emptyset$. Moreover, each vertex in $\overline{K}$ must be adjacent in G to the endvertex of a path on 363 vertices in $G[K]$. Consequently, $\overline{K} \subset \{Z_i, Z_i' : i = 1, \ldots, 6\}$. We may assume, without loss of generality that $Z_1 \in \overline{K}$. We may also assume that the path in $G[K]$ on 363 vertices that qualifies Z_1 for inclusion in $\overline{K}$ is $\Pi_{1,6} - Z_1$. Then, by the structure of the longest paths as determined in Lemma 2.1, $Z_5, Z_5' \in K$. Consequently, $Z_3, Z_3' \in K$ as for either of Z_3 and Z_3' to be in $\overline{K}$ requires $Z_1, Z_1' \in K$. But this means that $\Pi_{5,6}$ is a path on 364 vertices in $G[K]$ contradicting our choice of K as a P_{364}-kernel. This contradiction completes the proof of the theorem. $\square$

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